

DUNKL HARMONIC ANALYSIS AND FUNDAMENTAL SETS OF CONTINUOUS FUNCTIONS ON THE UNIT SPHERE

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Abstract. We establish a necessary and sufficient condition on a continuous function on $[-1, 1]$ under which the family of functions on the unit sphere \mathbb{S}^{d-1} constructed in the described manner is fundamental in $C(\mathbb{S}^{d-1})$. In our construction of functions and proof of the result, we essentially use Dunkl harmonic analysis.

Key words and phrases: fundamental set, continuous function, unit sphere, Dunkl intertwining operator, κ -spherical harmonics

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1. Introduction and preliminaries

We need some elements of the general Dunkl theory (see [2–6]); for a background on reflection groups and root systems the reader is referred to [6, 7].

Let \mathbb{R}^d denote d -dimensional Euclidean space. For $x \in \mathbb{R}^d$, we write $x = (x_1, \dots, x_d)$. The inner product of $x, y \in \mathbb{R}^d$ is denoted by $\langle x, y \rangle = \sum_{i=1}^d x_i y_i$, and the norm of x is denoted by $\|x\| = \sqrt{\langle x, x \rangle}$.

The unit sphere \mathbb{S}^{d-1} , $d \geq 2$, and the unit ball \mathbb{B}^d of \mathbb{R}^d are defined by

$$\mathbb{S}^{d-1} = \{x: \|x\| = 1\} \quad \text{and} \quad \mathbb{B}^d = \{x: \|x\| \leq 1\}.$$

For a nonzero vector $v \in \mathbb{R}^d$, define the reflection σ_v by

$$\sigma_v(x) = x - 2 \frac{\langle x, v \rangle}{\|v\|^2} v, \quad x \in \mathbb{R}^d.$$

Each reflection σ_v is contained in the orthogonal group $O(\mathbb{R}^d)$.

We give some basic definitions and notions which will be important.

Definition 1. Let $R \subset \mathbb{R}^d \setminus \{0\}$ be a finite set. Then R is called a root system if

- (1) $R \cap \mathbb{R}v = \{\pm v\}$ for all $v \in R$;
- (2) $\sigma_v(R) = R$ for all $v \in R$.

The subgroup $G = G(R) \subset O(\mathbb{R}^d)$ which is generated by the reflections $\{\sigma_v: v \in R\}$ is called the reflection group associated with R .

For any root system R in \mathbb{R}^d , the reflection group $G = G(R)$ is finite. The set of reflections contained in $G(R)$ is exactly $\{\sigma_v: v \in R\}$.

Each root system can be written as a disjoint union $R = R_+ \cup -R_+$, where R_+ and $-R_+$ are separated by a hyperplane through the origin. Such a set R_+ is called a positive subsystem. Its choice is not unique.

Definition 2. A nonnegative function κ on a root system R is called a multiplicity function on R if it is G -invariant, i.e. $\kappa(v) = \kappa(g(v))$ for all $v \in R$, $g \in G$.

Definition 3. The Dunkl operators are defined by

$$\mathcal{D}_i f(x) = \frac{\partial f(x)}{\partial x_i} + \sum_{v \in R_+} \kappa(v) \frac{f(x) - f(\sigma_v(x))}{\langle x, v \rangle} \langle v, e_i \rangle, \quad 1 \leq i \leq d,$$

where e_1, \dots, e_d are the standard unit vectors of \mathbb{R}^d .

The above definition does not depend on the special choice of R_+ , thanks to the G -invariance of κ . In case $\kappa = 0$, the Dunkl operators reduce to the corresponding partial derivatives.

Suppose Π^d is the space of all polynomials in d variables with complex coefficients, \mathcal{P}_n^d ($n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$) is the subspace of homogeneous polynomials of degree n in d variables.

According to [5], there exists a unique linear isomorphism V_κ of Π^d such that

$$V_\kappa(\mathcal{P}_n^d) = \mathcal{P}_n^d, \quad n \in \mathbb{N}_0, \quad V_\kappa 1 = 1, \quad \text{and} \quad \mathcal{D}_i V_\kappa = V_\kappa \frac{\partial}{\partial x_i}, \quad 1 \leq i \leq d.$$

This operator is called the Dunkl intertwining operator. If $\kappa = 0$, V_κ becomes the identity operator.

M. Rösler has proved in [8] that for each $x \in \mathbb{R}^d$ there exists a unique probability measure μ_x^κ on the Borel σ -algebra of \mathbb{R}^d with support in $\{y: \|y\| \leq \|x\|\}$, such that for all polynomials p on \mathbb{R}^d we have

$$V_\kappa p(x) = \int_{\mathbb{R}^d} p(y) d\mu_x^\kappa(y). \quad (1)$$

For $X = [-1, 1]$ or $X = \mathbb{S}^{d-1}$, we denote by $C(X)$ the space of continuous complex-valued functions on X .

Definition 4. We define the Dunkl truncated intertwining operator at an arbitrary point $\xi \in \mathbb{S}^{d-1}$

$$V_\kappa(\xi): C([-1, 1]) \rightarrow C(\mathbb{S}^{d-1})$$

by

$$V_\kappa(\xi; g, x) = \int_{\mathbb{R}^d} g(\langle \xi, \zeta \rangle) d\mu_x^\kappa(\zeta), \quad x \in \mathbb{S}^{d-1}, \quad g \in C[-1, 1],$$

where μ_x^κ is the measure given in (1).

The operator $V_\kappa(\xi)$ is well defined. Indeed, there exists a sequence $\{p_n\}$ of polynomials in Π^1 such that

$$\sup_{t \in [-1, 1]} |g(t) - p_n(t)| \rightarrow 0, \quad n \rightarrow \infty.$$

Note that $p_n(\langle \xi, \cdot \rangle) \in \Pi^d$. Thus, for all $x \in \mathbb{B}^d$,

$$\begin{aligned} |V_\kappa(\xi; g, x) - V_\kappa[p_n(\langle \xi, \cdot \rangle)](x)| &\leq \int_{\mathbb{B}^d} |g(\langle \xi, \zeta \rangle) - p_n(\langle \xi, \zeta \rangle)| d\mu_x^\kappa(\zeta) \\ &\leq \sup_{\zeta \in \mathbb{B}^d} |g(\langle \xi, \zeta \rangle) - p_n(\langle \xi, \zeta \rangle)| = \sup_{t \in [-1, 1]} |g(t) - p_n(t)| \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

As $V_\kappa[p_n(\langle \xi, \cdot \rangle)]$ is continuous on \mathbb{R}^d , then we deduce that $V_\kappa(\xi; g)$ is continuous on the unit ball \mathbb{B}^d .

In the present paper, we establish a necessary and sufficient condition on a function $g \in C([-1, 1])$ under which the family of functions $\{V_\kappa(x; g) : x \in \mathbb{S}^{d-1}\}$ is fundamental in $C(\mathbb{S}^{d-1})$. This result generalizes Theorem 2 in [9]. We state and prove the main result in section 4.

Recall that a set \mathcal{F} in a Banach space \mathcal{E} is said to be fundamental if the linear span of \mathcal{F} is dense in \mathcal{E} .

2. Fundamentality in $C(\mathbb{S}^{d-1})$

Suppose that the unit sphere \mathbb{S}^{d-1} is equipped with a positive Borel measure and notions of orthogonality for functions on \mathbb{S}^{d-1} are defined in terms of this Borel measure.

Let there be given an orthogonal sequence $\{U_n\}_{n=1}^\infty$ of finite-dimensional subspaces in $C(\mathbb{S}^{d-1})$. It is assumed that $\bigcup_{n=1}^\infty U_n$ is fundamental in the space $C(\mathbb{S}^{d-1})$. For each n , let $\{u_{nj} : 1 \leq j \leq \dim U_n\}$ denote a real-valued orthonormal basis of U_n .

Assume that we possess a summability method, given by an infinite matrix $A = [A_{nm}]_{n,m=1}^\infty$ with complex entries that has these properties:

- (i) each row of A has only finitely many nonzero elements;
- (ii) $\lim_{n \rightarrow \infty} A_{nm}$ exists for each $m = 1, 2, \dots$;
- (iii) the sequence of functions $f_n(x, y) = \sum_m A_{nm} \sum_j u_{mj}(x) u_{mj}(y)$, where $x, y \in \mathbb{S}^{d-1}$, converges (as $n \rightarrow \infty$) uniformly in x and y to a limit function $f(x, y)$.

Theorem 1. *Suppose that the hypotheses given above are satisfied, and let f be as in (iii). In order that the set of functions $\{x \mapsto f(x, y) : y \in \mathbb{S}^{d-1}\}$ be fundamental in $C(\mathbb{S}^{d-1})$ it is necessary and sufficient that $\lim_{n \rightarrow \infty} A_{nm}$ be nonzero for all m .*

This theorem was proved in [9, Section 2, Theorem 1] in a more general setting.

3. Some facts of Dunkl harmonic analysis on the unit sphere

The Dunkl Laplacian is defined by

$$\Delta_\kappa = \mathcal{D}_1^2 + \dots + \mathcal{D}_d^2.$$

The Dunkl Laplacian plays the role of the ordinary Laplacian. In the special case $\kappa = 0$, Δ_κ reduces to the ordinary Laplacian.

A κ -harmonic polynomial P of degree $n \in \mathbb{N}_0$ is a homogeneous polynomial $P \in \mathcal{P}_n^d$ such that $\Delta_\kappa P = 0$. The κ -spherical harmonics of degree n are the restriction of κ -harmonics of degree n to the unit sphere \mathbb{S}^{d-1} . Let $\mathcal{A}_n^d(\kappa)$ be the space of κ -spherical harmonics of degree n and let $N(n, d)$ be the dimension of $\mathcal{A}_n^d(\kappa)$.

The weighted inner product of $f, h \in C(\mathbb{S}^{d-1})$ is denoted by

$$\langle f, h \rangle_\kappa = \frac{1}{\sigma_d^\kappa} \int_{\mathbb{S}^{d-1}} f(x) \overline{h(x)} w_\kappa(x) d\omega(x),$$

where $d\omega$ is the Lebesgue measure on \mathbb{S}^{d-1} , w_κ is the weight function, invariant under the reflection group G , defined by

$$w_\kappa(x) = \prod_{v \in R_+} |\langle v, x \rangle|^{2\kappa(v)}, \quad x \in \mathbb{S}^{d-1},$$

and σ_d^κ is the constant chosen such that $\langle 1, 1 \rangle_\kappa = 1$.

Note that if $\kappa = 0$, then $w_\kappa = 1$.

In the rest of this section, we assume that $\kappa \neq 0$ if $d = 2$.

The following properties hold:

(I) $\bigcup_{n=0}^{\infty} \mathcal{A}_n^d(\kappa)$ is fundamental in $C(\mathbb{S}^{d-1})$.

(II) For any real-valued orthonormal basis $\{S_{n,j}^\kappa: 1 \leq j \leq N(n, d)\}$ of $\mathcal{A}_n^d(\kappa)$,

$$V_\kappa(x; C_n^{\lambda_\kappa}, y) = \frac{\lambda_\kappa}{n + \lambda_\kappa} \sum_{j=1}^{N(n, d)} S_{n,j}^\kappa(x) S_{n,j}^\kappa(y), \quad x, y \in \mathbb{S}^{d-1},$$

where $C_n^{\lambda_\kappa}(\cdot)$ denotes the Gegenbauer polynomial, λ_κ is a positive constant defined by

$$\lambda_\kappa = \sum_{v \in R_+} \kappa(v) + \frac{d-2}{2} \quad (2)$$

(see, for example, [2, Section 7.2]).

(III) If $n \neq m$, then $\mathcal{A}_n^d(\kappa) \perp \mathcal{A}_m^d(\kappa)$, i.e. $\langle P, Q \rangle_\kappa = 0$ for $P \in \mathcal{A}_n^d(\kappa)$ and $Q \in \mathcal{A}_m^d(\kappa)$ [3, Theorem 1.6].

Property (I) follows from the Weierstrass approximation theorem: if f is continuous on \mathbb{S}^{d-1} , then it can be uniformly approximated by polynomials restricted to \mathbb{S}^{d-1} . According to [3, Theorem 1.7], these restrictions belong to the linear span of $\bigcup_{n=0}^{\infty} \mathcal{A}_n^d(\kappa)$.

4. Main result and its proof

We can now establish the main result of the paper.

Theorem 2. *Fix $d \geq 2$. Suppose R is a fixed root system in \mathbb{R}^d , κ is a multiplicity function on R . Assume that $\kappa \neq 0$ if $d = 2$. Let $g \in C([-1, 1])$. In order that the family of functions $\{V_\kappa(x; g) : x \in \mathbb{S}^{d-1}\}$ be fundamental in $C(\mathbb{S}^{d-1})$ it is necessary and sufficient that*

$$\int_{-1}^1 g(t) C_n^{\lambda_\kappa}(t) (1-t^2)^{\lambda_\kappa-1/2} dt \neq 0, \quad n = 0, 1, 2, \dots,$$

where the constant λ_κ is defined in (2).

P r o o f. For $\lambda > 0$, let

$$c_\lambda = \left(\int_{-1}^1 (1-t^2)^{\lambda-1/2} dt \right)^{-1}$$

be the normalizing constant. Then the Gegenbauer expansion of g takes the form

$$g(t) \sim \sum_{n=0}^{\infty} b_n \frac{n+\lambda}{\lambda} C_n^\lambda(t) \quad \text{with} \quad b_n = \frac{c_\lambda}{C_n^\lambda(1)} \int_{-1}^1 g(t) C_n^\lambda(t) (1-t^2)^{\lambda-1/2} dt, \quad (3)$$

since $c_\lambda \int_{-1}^1 (C_n^\lambda(t))^2 (1-t^2)^{\lambda-1/2} dt = C_n^\lambda(1) \lambda / (n+\lambda)$.

For $\delta > 0$, the Cesàro (C, δ) means of the above series are

$$S_n^\delta g(t) = \frac{1}{A_n^\delta} \sum_{m=0}^n A_{n-m}^\delta b_m \frac{m+\lambda}{\lambda} C_m^\lambda(t), \quad A_n^\delta = \binom{n+\delta}{n}. \quad (4)$$

Note that $A_{n-m}^\delta / A_n^\delta \rightarrow 1$ as $n \rightarrow \infty$ for each m .

If $\delta > \lambda$, then it follows from [1, Theorem 1.3] that the sequence $\{S_n^\delta g\}$ converges uniformly on $[-1, 1]$ to the function g .

Let $\lambda = \lambda_\kappa$ and $\delta > \lambda$. Then the sequence $\{V_\kappa(x; S_n^\delta g, y)\}$ converges uniformly in $x, y \in \mathbb{S}^{d-1}$ to $V_\kappa(x; g, y)$. Indeed,

$$\begin{aligned} & \sup_{x, y \in \mathbb{S}^{d-1}} |V_\kappa(x; g, y) - V_\kappa(x; S_n^\delta g, y)| \\ &= \sup_{x, y \in \mathbb{S}^{d-1}} \left| \int_{\mathbb{B}^d} g(\langle x, \xi \rangle) d\mu_y^\kappa(\xi) - \int_{\mathbb{B}^d} S_n^\delta g(\langle x, \xi \rangle) d\mu_y^\kappa(\xi) \right| \\ &\leq \sup_{x, \xi \in \mathbb{S}^{d-1}} |g(\langle x, \xi \rangle) - S_n^\delta g(\langle x, \xi \rangle)| \\ &= \sup_{t \in [-1, 1]} |g(t) - S_n^\delta g(t)| \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

Using Property (II) of the κ -spherical harmonics (see Section 3) and (4), we get

$$V_\kappa(x; g, y) = \lim_{n \rightarrow \infty} \sum_{m=0}^n \frac{A_{n-m}^\delta}{A_n^\delta} b_m \sum_{j=1}^{N(m, d)} S_{m,j}^\kappa(x) S_{m,j}^\kappa(y).$$

We can apply Theorem 1 with $U_n = \mathcal{A}_n^d(\kappa)$, $u_{nj} = S_{n,j}^\kappa$, and $A_{nm} = (A_{n-m}^\delta b_m)/A_n^\delta$ for $m \leq n$, $A_{nm} = 0$ for $m > n$, to conclude that the condition

$$\lim_{n \rightarrow \infty} \left(\frac{A_{n-m}^\delta}{A_n^\delta} b_m \right) \neq 0 \quad \text{for each } m$$

is the necessary and sufficient condition for fundamentality. Obviously, this condition reduces to $b_m \neq 0$ for all m . By (3), the latter is equivalent to the integral condition described in the theorem. \square

The approach used in the proof of this theorem is as in [9, Theorem 2].

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